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THE EXTREMAL PROPERTY OF THE CONSTRAINT REACTIONS OF SYSTEMS OF CONNECTED BODIES†

A. P. LEUTIN

Zhukovskii

e-mail: filatyev@tsagi.ru

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The extremal property of the forces and moments of reactions in the joints of mechanical systems consisting of a carrier and bodies connected to it is investigated, based on the principle of least constraint. The problem of minimizing the functional of the linear and angular accelerations of bodies, taking into account holonomic retaining constraints in which there is active friction is solved by the method of undetermined multipliers. It is shown that their values for the actual motion of the bodies coincide with the forces and moments of the constraint reactions, and the least constraint is determined. The results obtained are interpreted from the positions of elasticity theory. The separation of the stages of rockets is considered on the basis of the principle of least constraint.

A well-known example of a system consisting of a central body and peripheral bodies connected to it is the Soyuz carrier rocket four boosters are attached sideways-on to a central stage and, in the jettison process, they are released from the side. The jettison of the side boosters from the central part of the Energiya rocket, the release of suspended loads from carrier aircraft, etc., occur in a similar way [1, 2].

1. BASIC RELATIONS OF GAUSS' PRINCIPLE

According to Gauss' principle [3, 4], the actual motion of a system with ideal constraints can be distinguished among those kinematically possible (constraint compatible) in that, at each instant of time, the least value of the constraint Z is achieved on it, comprising the weighted sum of the squares of the deviations of the accelerations of point masses of the system from their accelerations when there are no constraints (in free motion):

$$Z = \frac{1}{2}\sum_{n=1}^{N} m_n\left(\dot{\mathbf{V}}_n - \frac{\mathbf{F}_{an}}{m_n}\right)^2 \tag{1.1}$$

where m_n is the mass of the nth point mass of the system ($n = 1, \ldots, N$) $\dot{\mathbf{V}}_n$ is the vector of absolute acceleration of the point, and $\mathbf{F}_{an}$ is the prescribed vector of the active external force acting at the nth point. The significance of Gauss' principle is that a non free system executes motion that is closed to free motion, which is considered to be known; the constraint (1.1) is a measure of this closeness.

Along with the formulation of Gauss' principle, based on a comparison of the constrained and free motions of bodies of the system, its force interpretation is also possible. Since $m_n\dot{\mathbf{V}}_n = \mathbf{F}_{an} + \mathbf{R}_n$, where $\mathbf{R}_n$ is the total force of the constraint reaction of the nth point mass, it follows that

$$Z = \frac{1}{2}\sum_{n=1}^{N}\frac{\mathbf{R}_n^2}{m_n} \tag{1.2}$$

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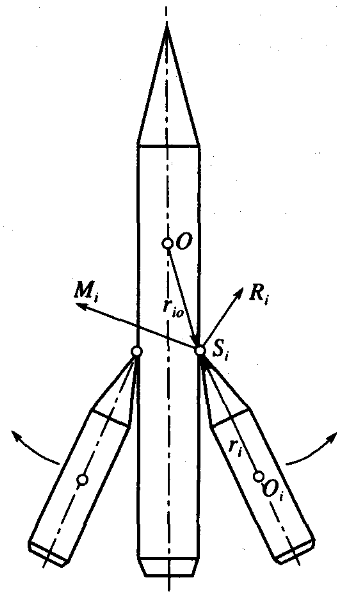

Fig. 1

Thus, the principle of lease constraint for actual motion of a system of point masses leads to an extremal property of the reactions – the functional (1.2) at any instant of time reaches a conditional minimum (satisfying the constraints imposed on the system).

When some of the constraints are removed, the least value of the constraint of the system decreases. With the point masses, completely released, when the accelerations are determined solely by the applied active forces, an unconditional least constraint is achieved: min $Z = 0$.

The constraint of the system of rigid bodies contains, in particular, the constraints of the individual bodies [5, 6]

$$Z = S - \dot{\mathbf{V}}\mathbf{F}_a - \dot{\boldsymbol{\omega}}\mathbf{M}_a + \dots$$
$$S = \frac{1}{2}\sum_{n=1}^{N} m_n \dot{\mathbf{v}}_n^2 = \frac{1}{2}(m\dot{\mathbf{V}}^2 + \dot{\boldsymbol{\omega}}I\dot{\boldsymbol{\omega}}) + \dot{\boldsymbol{\omega}}(\boldsymbol{\omega} \times I\boldsymbol{\omega}) + \dots \tag{1.3}$$

when S is the energy of accelerations of the body, defined in the same way as the kinetic energy of the body with the velocities replaced by accelerations, m is the mass of the body, I is the matrix of the moments of inertia of the body about its centre of mass in the axes connected to it, $\dot{\mathbf{V}}_n$ is the vector of absolute acceleration of the centre of mass, and $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the vectors of the absolute angular velocity and acceleration of the body: $\mathbf{F}_a$ and $\mathbf{M}_a$ are the principal vector and principal moment (about the centre of mass) of the active external forces applied to the body; the dots denote terms which depend on the accelerations caused by the active forces and moments, and also on the angular velocities of the body, which are considered to be given. The first two terms in the relation for the energy of the accelerations correspond to Koenig's formula for the kinetic energy of a body with the velocities of the points replaced by the accelerations corresponding to them [3]. The terms $\dot{\mathbf{V}}\mathbf{F}_a + \dot{\boldsymbol{\omega}}\mathbf{M}_a$ comprise the acceleration-dependent part of the power of the active forces and moments in displacements of the body as part of the system, where both active forces and moments and also reactions of the constraints with neighbouring bodies and friction forces act on it.

2. THE CONSTRAINT FOR A "CARRIER PLUS LOADS" SYSTEM

Consider a system of bodies with the "tree" structure that consists of a central carrier and an arbitrary number L of loads (see Fig. 1). The constraints are assumed to be holonomic, retaining, and non-ideal: friction acts in them. Special cases are the sliding of a load without rotation or rotation without sliding and, finally, relative rest [1]. Below, friction is nominally considered to be an active factor, i.e. the forces and moments of friction are presumed, like other active forces and moments, to be continuous when the bodies are released from constraints. This makes it possible to reduce the problem to the determination of the motion of a system with ideal constraints and, ultimately, to the minimization of the constraint functional subject to restrictions (the action of friction is then determined using iterations [2]).

The accelerations of the bodies of the system can be written in the form

$$\begin{aligned} \dot{\mathbf{V}}_{ia} &= \dot{\mathbf{V}}_i + \Delta\dot{\mathbf{V}}_i, \quad \dot{\boldsymbol{\omega}}_{ia} = \dot{\boldsymbol{\omega}}_i + \Delta\dot{\boldsymbol{\omega}}_i \\ \dot{\mathbf{V}}_{a0} &= \dot{\mathbf{V}}_0 + \Delta\dot{\mathbf{V}}_0, \quad \dot{\boldsymbol{\omega}}_{a0} = \dot{\boldsymbol{\omega}}_0 + \Delta\dot{\boldsymbol{\omega}}_0 \end{aligned} \tag{2.1}$$

The subscript a denotes accelerations of the centres of mass of the bodies and their angular accelerations caused by the active forces and moments. The vectors

$$\Delta\dot{\mathbf{V}}_i = -m_i^{-1}\mathbf{R}_i, \quad \Delta\dot{\mathbf{V}}_0 = m_0^{-1}\sum_{i=1}^{L}\mathbf{R}_i$$

$$\Delta\dot{\boldsymbol{\omega}}_i = -I_i^{-1}(\mathbf{M}_i + \mathbf{r}_i \times \mathbf{R}_i), \quad \Delta\dot{\boldsymbol{\omega}}_0 = I_0^{-1}\sum_{i=1}^{L}(\mathbf{M}_i + \mathbf{r}_{i0} \times \mathbf{R}_i)$$

are increments of the accelerations of the loads and the carrier that are acquired by them after being released owing to termination of the action of the total forces $\mathbf{R}_i$ and moments $\mathbf{M}_i$ of the constraint reactions; $\mathbf{r}_i$ and $\mathbf{r}_{i0}$ are the radius vectors of the poles of the joints S_i, drawn from the centres of mass of the load O_i and the carrier O. It is assumed that the forces $\mathbf{R}_i$ and moments $\mathbf{M}_i$ of the reactions correspond to the action of the carrier on the loads. The quantities relating to the carrier and the ith load are given the subscripts 0 and i ($i = 1, \dots, L$).

Using relation (1.3) obtain that expression (1.1) for the constraint takes the form

$$\begin{aligned} Z &= \Delta Z + Z_a \\ \Delta Z &= \frac{1}{2}(m_0\Delta\dot{\mathbf{V}}_0^2 + \Delta\dot{\boldsymbol{\omega}}_0 I_0 \Delta\dot{\boldsymbol{\omega}}_0) + \frac{1}{2}\sum_{i=1}^{L}(m_i\Delta\dot{\mathbf{V}}_i^2 + \Delta\dot{\boldsymbol{\omega}}_i I_i \Delta\dot{\boldsymbol{\omega}}_i) \end{aligned} \tag{2.2}$$

where ΔZ is the variable part of the constraint, the least value of which is attained for actual motion of the system. This part of the constraint is equal to the energy of the accelerations of the bodies and they acquire on complete release. In fact, in this case the accelerations of points of each body change by $\Delta\dot{\mathbf{v}}_n = \Delta\dot{\mathbf{V}} + \Delta\dot{\boldsymbol{\omega}} \times \mathbf{r}_n$, where $\mathbf{r}_n$ is the radius vector of the nth point of the body with respect to its centre of mass. Then, from the formula given in [5] for the energy of accelerations of bodies, we obtain expression (2.2). When the bodies are completely released $\Delta Z = 0$. The second part of the constraint, Z_a, depends only on the prescribed angular velocities and accelerations of the bodies, caused by the active forces and moments. Therefore, it is not varied and will not be considered below.

3. MINIMIZATION OF THE CONSTRAINT TAKING THE CONSTRAINT CONDITIONS INTO ACCOUNT

When determining the actual accelerations of the connected bodies, we arrive at the problem of minimizing the quadratic functional ΔZ (2.2) when there are linear restrictions of the equality type. To solve it by the Lagrange method, we will introduce an extended functional, which takes into account the presence of constraints using undetermined multipliers

$$G = \Delta Z + \sum_{i=1}^{L}(\boldsymbol{\lambda}_{iv}\mathbf{f}_{iv} + \boldsymbol{\lambda}_{i\omega}\mathbf{f}_{i\omega}) \tag{3.1}$$

where $\mathbf{f}_{iv}$, $\boldsymbol{\lambda}_{iv}$, $\mathbf{f}_{i\omega}$, and $\boldsymbol{\lambda}_{i\omega}$ are vectors determining the conditions of constraint of the loads with the carrier, and the multipliers corresponding to them ($i = 1, \dots, L$). When writing the constraint conditions, the motion of the carrier with respect to the loads is regarded as translational.

The vector $\mathbf{f}_{i\upsilon}$ specifies the conditions for there to be no displacement of the pole of the joint S_i with respect to the carrier for the prescribed directions (across the guideline or the plane on the carrier) or, in a special case, the conditions for relative immobility of the pole

$$\mathbf{f}_{i\upsilon} = \Delta\dot{\mathbf{V}}_i + \Delta\dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i - \Delta\dot{\mathbf{V}}_0 - \Delta\dot{\boldsymbol{\omega}}_0 \times \mathbf{r}_{i0} - \delta\dot{\mathbf{v}}_{ia} = \mathbf{0} \tag{3.2}$$

where $\delta\dot{\mathbf{v}}_{ia} = \dot{\mathbf{v}}_{ia} - \dot{\mathbf{v}}_{a0} - \mathbf{w}_{ic}$ is the relative acceleration of the pole S_i caused by active forces and moments, which occurs when the constraint at a given unit disappears, and $\mathbf{w}_{ic}$ is the Coriolis acceleration of the pole.

The vector $\mathbf{f}_{i\omega}$ specifies the conditions for there to be no relative turning of the ith load perpendicular to its axis of rotation or the conditions for its relative rest:

$$\mathbf{f}_{i\omega} = \Delta\dot{\boldsymbol{\omega}}_i - \Delta\dot{\boldsymbol{\omega}}_0 - \delta\dot{\boldsymbol{\omega}}_{ia} = \mathbf{0} \tag{3.3}$$

where $\delta\dot{\boldsymbol{\omega}}_{ia} = \dot{\boldsymbol{\omega}}_{ia} - \dot{\boldsymbol{\omega}}_{a0} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_0$ is the relative angular acceleration of the load caused by active forces and moments.

From the necessary conditions for a minimum of functional (3.1) we obtain the relations between the increments of accelerations of the bodies and the undetermined multipliers

$$\Delta\dot{V}_i = -m_i^{-1} C_i^T B_{i\upsilon}\lambda_{i\upsilon}, \quad \Delta\dot{V}_0 = m_0^{-1}\sum B_{i\upsilon}\lambda_{i\upsilon} \tag{3.4}$$

$$\begin{aligned} \Delta\dot{\omega}_i &= -I_i^{-1}(\tilde{r}_i C_i^T B_{i\upsilon}\lambda_{i\upsilon} + C_i^T B_{i\omega}\lambda_{i\omega}) \\ \Delta\dot{\omega}_0 &= I_0^{-1}\sum(\tilde{r}_{i0} B_{i\upsilon}\lambda_{i\upsilon} + B_{i\omega}\lambda_{i\omega}) \end{aligned} \tag{3.5}$$

In matrix notation, the vectors are specified as columns of coordinates in systems rigidly connected to the bodies. The orthogonal matrix C_i specifies the transformation of coordinates from axes of the carrier to axes of the ith load. The constraint conditions (3.2) and (3.3) of the loads with the carrier are projected onto the corresponding directions using the matrices $B_{i\upsilon}$ and $B_{i\omega}$ (they are made up of the coordinates of the unit vectors of the axes, along which there are no relative displacements and rotations of the given load [2]; depending on the type of relative motion of the loads, the matrices have dimensions 3×1, 3×2 or 3×3). Here and below, $\tilde{a}$ is a skew-symmetric matrix that can be used to write the vector products with vector $\mathbf{a}$.

As can be seen, relations (3.4) and (3.5) are the equations of motion of the carrier and loads [see Eq. (2.1)], where, instead of the vectors of the total forces R_i and moments M_i of the reactions expressed in axes onto which the constraint conditions are projected, there are unknown multipliers $\lambda_{i\upsilon}$ and $\lambda_{i\omega}$ respectively. Substituting expressions (3.4) and (3.5) into the constraint conditions (2.4) and (2.5), we obtain, for extremal values of multipliers $\lambda_{i\upsilon}^*$ and $\lambda_{i\omega}^*$ corresponding to the actual motion of the bodies, the same system of equations as that obtained by direct calculation of the forces and moments of the reactions directly from the constraint conditions [1, 2]

$$Ax^* = b \tag{3.6}$$

where $A = ||A_{ij}||$ is the partitioned matrix of the system $(i, j = 1, \ldots, L)$, $x^* = ||x_i^*||$ is the combined column vector of the extremal values of the reactions and $b = ||b_i||$ is the column of the right-hand sides of the equations specifying the constraint conditions of the bodies, where

$$b_i = -\left\| \begin{matrix} B_{i\upsilon}^T \, \delta\dot{\upsilon}_{ia} \\ B_{i\omega}^T \, \delta\dot{\omega}_{ia} \end{matrix} \right\|, \quad x_i^* = \left\| \begin{matrix} R_i^* \\ M_i^* \end{matrix} \right\| = \left\| \begin{matrix} \lambda_{i\upsilon}^* \\ \lambda_{i\omega}^* \end{matrix} \right\|$$

The vectors $\delta\dot{\upsilon}_{ia}$, $\delta\dot{\omega}_{ia}$ are given in axes of the carrier.

The diagonal symmetric submatrices

$$A_{ii} = \left\| \begin{matrix} A'_{ii} & B_{ii} \\ B_{ii}^T & D_{ii} \end{matrix} \right\|$$

define the interaction via the constraint unit of the ith load with the carrier

$$A'_{ii} = \tilde{m}_i^{-1}E_i - B_{iv}^T(C_i\tilde{r}_iI_i^{-1}\tilde{r}_iC_i^T + \tilde{r}_{r0}I_0^{-1}\tilde{r}_{i0})B_{iv}$$
$$B_{ii} = -B_{iv}^T(C_i\tilde{r}_iI_i^{-1}C_i^T + \tilde{r}_{i0}I_0^{-1})B_{i\omega}$$
$$D_{ii} = B_{i\omega}^T(C_iI_i^{-1}C_i^T + I_0^{-1})B_{i\omega}$$

where $\tilde{m}_i = m_im_0/(m_i + m_0)$ is the reduced mass of the carrier and the ith load, and E_i is the unit matrix corresponding in dimensionality to the vector of the reaction force at the ith unit (in particular, the matrix can degenerate into ordinary unity). The non-diagonal partitioned matrices

$$A_{ij} = \left\| \begin{matrix} T_{ij} & Q_{ij} \\ W_{ij} & P_{ij} \end{matrix} \right\|$$

determine the action of the jth load on the ith load ($j \neq i$) which occurs via the support

$$T_{ij} = m_0^{-1}B_{iv}^TB_{jv} - B_{iv}^T\tilde{r}_{r0}I_0^{-1}\tilde{r}_{j0}B_{jv}, \quad Q_{ij} = -B_{iv}^T\tilde{r}_{i0}I_0^{-1}B_{j\omega}$$
$$P_{ij} = B_{i\omega}^TI_0^{-1}B_{j\omega}, \quad W_{ij} = Q_{ji}^T$$

In the general case, the matrix A_{ij} is not quadratic since the dimensions of the vectors of the reaction forces and moments at the different units are dissimilar. From the structure of the submatrices it follows that $A_{ij}^T = A_{ji}$, i.e. the matrix A is quadratic and symmetric (and also positive-definite – see below). In the simplest case of a single load, only a diagonal matrix, $i = L = 1$, remains in A.

Note that, when the constraint equations of the bodies are written in the form (3.2), (3.3), the extremal values of the Lagrange multipliers are identical with the actual values of the forces and moments of the constraint reactions.

As pointed out above, the right-hand side of the linear system of equations (3.6) contains the friction forces and moments, which depend non-linearly on the reaction forces and moments. The equations is therefore solved by iteration. At the first step, the constraints are assumed to be ideal – without friction. At the following steps, the friction forces and moments in the constraints are calculated from the values of the reaction and moments, determined at the previous step. As a result, there is a change in the original "ideal" reactions, caused by the action of friction. Numerical modelling of the motion of real "carrier plus loads" systems indicates that, for solid antifriction coatings (Coulomb friction) used in the joints of aerospace systems, iteration processes converge fairy rapidly: an accuracy of ~1% is achieved after 3–5 iterations [2].

4. THE FUNCTIONAL WHICH DEPENDS ON THE REACTION FORCES AND MOMENTS

On the basis of the extended functional (3.1) for the constraint, it is possible to obtain the functional of the reaction forces and moments, which can be interpreted from standpoints of elasticity theory. To do this, we substitute the extremal values of the multipliers λ_{iv}^* and $\lambda_{i\omega}^*$ into (3.1) and express the accelerations acquired by the bodies when released in terms of the variable values of the reaction forces $\mathbf{R}_i$ and moments $\mathbf{M}_i$ (2.1). After identity transformations we obtain the quadratic functional

$$G(x) = \frac{1}{2}x^TAx + \lambda^{*T}f(x) = \frac{1}{2}x^TAx + b^TA^{-1}(-Ax + b) = \frac{1}{2}x^TAx - b^Tx + \dots \tag{4.1}$$

where $\lambda^* = ||\lambda_i^*||$ is the combined column vector of extremal values of the Lagrange multipliers corresponding to the constraint conditions (3.3) and (3.4), $x = ||x_i||$ is the column vector defining the reaction forces and moments, and $f(x) = ||f_i||$ is the vector of the left-hand sides of the constraint equations of all the bodies of the system, where

$$\lambda_i^* = \left\| \begin{matrix} \lambda_{iv}^* \\ \lambda_{i\omega}^* \end{matrix} \right\|, \quad x_i = \left\| \begin{matrix} R_i \\ M_i \end{matrix} \right\|, \quad f_i = \left\| \begin{matrix} f_{iv} \\ f_{i\omega} \end{matrix} \right\|$$

Equating to zero the derivative with respect to the variable vector x, we obtain the well-known solution $x^* = A^{-1}b$. On the other hand, the functional (4.1) is formally obtained from the minimum condition of the (3.6) by integrating it with respect to the variable x.

Thus, the extremal property of the reactions of the constraints of the "carrier-plus-loads" system consists of the fact that, during its actual motion, at each instant of time, values of the reaction forces and moments $\mathbf{R}_i^*$ and $\mathbf{M}_i^*$ at all units of the constraint ($i = 1, \dots, L$) are realized that make the functional of the constraint ΔZ a conditional minimum and the extended functional G an unconditional minimum, and here these forces and moments are identical with the corresponding extremal values of the Lagrange multipliers. As a result, the least value of the constraint functional is obtained

$$\min\Delta Z = \frac{1}{2}x^{*T}Ax^* = \frac{1}{2}b^TA^{-1}b \tag{4.2}$$

This quantity can also be calculated using the equality $\lambda^{*T} = d(\min \Delta Z)/db$, known from optimization theory [7]. Then

$$\min\Delta Z = \int_0^b \lambda^{*T} db = \frac{1}{2} b^TA^{-1}b$$

The value of the constraint obtained is a measure of the closeness of the constrained and free motions of the system of the bodies and is realized with the instantaneous disappearance of the constraints (for example, when the separation of the stages of aircraft occurs, see Section 5). From the positiveness of $\min \Delta Z$ there follows the positive definiteness of the matrices A and A^{-1}; this guarantees the uniqueness of the solution of Eqs (3.6) and, consequently, the uniqueness of motion of the systems of bodies.

The relations obtained above are similar in structure to the main relations of elasticity theory [8, 9]. The scalar product in (4.1)

$$-b^Tx = \sum(R_i^TB_{iv}^T\delta\dot{v}_{ia} + M_i^TB_{i\omega}^T\delta\dot{\omega}_{ia})$$

comprises the power of the reaction forces and moments in displacements of the loads with respect to the carrier, caused by the active forces and moments at the corresponding units of the constraint. The positive-definite quadratic from $x^TAx/2$ can be regarded as the potential energy of the forces and moments of the constraint reactions of the bodies. In fact, the energy of ideally elastic deformations of the structure is equal to $U = Q^TCQ/2$, where Q is the combined vector of the active external forces and moments (generalized forces) acting on the structure and C is the influence matrix. Thus, it is possible to interpret relation (3.6) as an analogue of Castigliano's theorem defining the necessary condition for a minimum of the potential energy, from which the actual distribution of deformations is determined (the principle of minimum potential energy). According to this theorem, the partial derivative of the energy of deformation of an ideally elastic structure is equal in generalized force to the displacement of the point of application of the force along its direction: $\partial U/\partial Q = q^T$, where q is the combined vector of displacements corresponding to the prescribed generalized forces.

The least constraint of the system of bodies means that the potential energy of the constraint reactions in actual motion also takes the least possible value.

5. THE MOTION OF AIRCRAFT STAGES AFTER BEING RELEASED FROM THE CONSTRAINTS

As a rule, when the stages (parts) of the aircraft are separated, only their accelerations change, and the velocities remain unchanged [1]. In this case, to analyse the separation, it is natural to apply Gauss' principle (when the constraint are removed, friction at the units "disappears" and does not remain unchanged as conventionally assumed above).

Suppose that, at a certain instant of time, instantaneous removal of all the constraints between the carrier and the L loads, hitherto rigidly fastened in a single whole, occurs. Before removal, $\dot{\omega}_i^- = \dot{\omega}_0^-$, $\omega_i = \omega_0$ and accelerations of the centres of mass are connected via the kinematics of the system as a rigid body. Separation is normally carried out during controlled balanced flight: before removing the constraints, there is no rotation of the carrier – $\omega_0 = \dot{\omega}_0^- = 0$, as a result of which $\dot{\mathbf{V}}_i^- = \dot{\mathbf{V}}_0^-$.

Then, using formula (1.3) for the energy of accelerations of the body and representing the accelerations after removal of the constraints (denoted by a plus sign) in terms of the accelerations directly before their removal (the minus sign)

$$\dot{\mathbf{V}}_i^+ = \dot{\mathbf{V}}_i^- + \Delta\dot{\mathbf{V}}_i', \quad \dot{\boldsymbol{\omega}}_i^+ = \dot{\boldsymbol{\omega}}_0^- + \Delta\dot{\boldsymbol{\omega}}_i'$$
$$\dot{\mathbf{V}}_0^+ = \dot{\mathbf{V}}_0^- + \Delta\dot{\mathbf{V}}_0', \quad \dot{\boldsymbol{\omega}}_0^+ = \dot{\boldsymbol{\omega}}_0^- + \Delta\dot{\boldsymbol{\omega}}_0'$$

we obtain the relation between the energies of the accelerations of the system before and after removal of the constraints of the bodies

$$S^+ - S^- = \Delta S + m_0\dot{\mathbf{V}}_0^-\Delta\dot{\mathbf{V}}_0' + \sum m_i\dot{\mathbf{V}}_0^-\Delta\dot{\mathbf{V}}_i' \tag{5.1}$$

where

$$S = S_0(\dot{\mathbf{V}}_0, \dot{\boldsymbol{\omega}}_0, \boldsymbol{\omega}_0) + \sum S_i(\dot{\mathbf{V}}_i, \dot{\boldsymbol{\omega}}_i, \boldsymbol{\omega}_0)$$
$$\Delta S = \Delta Z(\Delta\dot{\mathbf{V}}_0', \Delta\dot{\boldsymbol{\omega}}_0', \Delta\dot{\mathbf{V}}_i', \Delta\dot{\boldsymbol{\omega}}_i')$$

S is the energy of the accelerations of the system of bodies and ΔS is the energy of the accelerations acquired by the bodies when the constraints are completely removed [see Eq. (2.2)].

From relation (2.1) we have the equality

$$m_0\Delta\dot{\mathbf{V}}_0' + \sum m_i\Delta\dot{\mathbf{V}}_i' = \mathbf{0}$$

denoting that the acceleration of the centre of mass of the system is independent of external forces and moments. Then, from equality (5.1) we obtain an analogue of Carnot's second theorem, according to which, on the instantaneous disappearance of the constraints, the incremental in the kinetic energy of the system is equal to the energy of the acquired velocities of the bodies [10].

$$S^+ - S^- = \Delta S$$

Since the process of separation of the stages comprises the controlled decomposition of a single system into the individual bodies, this relation can be interpreted as a peculiar transition of the potential energy accumulated the constraints before their breaking into the energy of acquired accelerations of the released bodies. After breaking, the constraint units, drawn out by the reaction forces, act in the same way as compressed springs, pushing the bodies away from each other [1].

When there is no friction at the constrain units, the energy of the accelerations acquired by the bodies when they are completely released from the constraints is equal to the value of the constraint of system (4.2) before the release from constraints: $\Delta S = \min\Delta Z$. For aircraft, this equality is satisfied approximately, the more accurately the less the contribution of friction, which in practice is reduced by using special materials.

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